

On the Geometry and Quantization of Symplectic Howe Pairs

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Abstract

We study the orbit structure and the geometric quantization of a pair of mutually commuting hamiltonian actions on a symplectic manifold. If the pair of actions fulfils a *symplectic Howe condition*, we show that there is a canonical correspondence between the orbit spaces of the respective moment images. Furthermore, we show that reduced spaces with respect to the action of one group are symplectomorphic to coadjoint orbits of the other group. In the Kähler case we show that the linear representation of a pair of compact Lie groups on the geometric quantization of the manifold is then equipped with a representation-theoretic Howe duality.

1 Introduction

In representation theory, Howe duality ([How89]) is well-known. In the model situation of a product $G_1 \times G_2$ of two compact connected Lie groups linearly represented on a finite dimensional complex vector space U ($\varrho : G_1 \times G_2 \rightarrow GL(U)$), we say that the representation satisfies the *Howe condition* or is equipped with a *Howe duality* if there is a subset \mathcal{D} of $\widehat{G_1}$, the set of equivalence classes of

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irreducible complex representations of G_1 , and an injective map $\Lambda : \mathcal{D} \rightarrow \widehat{G_2}$ such that

$$(*) \quad U \cong \bigoplus_{\alpha \in \mathcal{D}} V_\alpha \otimes W_{\Lambda(\alpha)},$$

where V_α represents a class α in \mathcal{D} and $W_{\Lambda(\alpha)}$ represents the class $\Lambda(\alpha)$. Denoting for $k = 1, 2$ the restriction of the representation ϱ to G_k by ϱ_k (as well as the corresponding Lie algebra representation), condition $(*)$ is equivalent to

$$(**) \quad Z_{\text{End}(U)}(\varrho_i(\mathcal{U}\mathfrak{g}_i)) = \varrho_j(\mathcal{U}\mathfrak{g}_j) \quad \text{for } i \neq j,$$

where $\mathcal{U}\mathfrak{g}$ is the universal enveloping algebra (over \mathbb{C}) of a Lie algebra \mathfrak{g} , and for a set $S \subseteq \text{End}(U)$, $Z_{\text{End}(U)}(S)$ is the centralizer of S in the endomorphisms of U . This equivalence is easily shown using standard facts from representation theory, as found, e.g., in [GW98]. Obviously, condition $(*)$ can be formulated for representations on a complex vector space U having countable dimension and -upon passing to an appropriate completion on the right hand side- for representations on Fréchet or more general topological vector spaces.

In the spirit of quantization/dequantization of physical and mathematical structures (going back to the correspondence principle of Niels Bohr, see [RN76]), attempts to find analogous notions in symplectic geometry have been made. This concerns analogues of commutants (G_γ in [KKS78], polar actions in [OR04]) and dual pair notions (as in [Wei83]), as well as the study of the orbit structure in particular cases like in, e.g., [Ada87]. In this article, we study the setting of commuting hamiltonian (proper) actions of two Lie groups on a symplectic manifold. We define in this situation a *symplectic Howe condition*, naturally corresponding to the above condition $(**)$, namely

$$Z_{C^\infty(M)}(\Phi_i^* C^\infty(\mathfrak{g}_i^*)) = \Phi_j^* C^\infty(\mathfrak{g}_j^*) \quad \text{for } i \neq j.$$

Here and in the sequel, the Lie algebra of a Lie group G will always be denoted by \mathfrak{g} and, for $A \subseteq C^\infty(M)$, $Z_{C^\infty(M)}(A) = \{f \in C^\infty(M) \mid \{f, a\} = 0 \forall a \in A\}$.

We analyse thoroughly its consequences on the orbit structure, obtaining results that refine those of [KKS78]. We then prove an orbit correspondence and show the conservation of integrality under the correspondence. Furthermore we show that the Marsden-Weinstein reduced spaces of the action of one group are equivariantly symplectomorphic to coadjoint orbits in the moment image of the other group action. To this setting we finally apply geometric quantization and obtain in the Kähler case a decomposition of the quantization of the initial symplectic manifold as in $(*)$. That is, given a pair of commuting actions by holomorphic transformation groups on a Kähler manifold satisfying the symplectic Howe condition, the linear action of the pair on the holomorphic quantization (the holomorphic sections of the quantizing holomorphic line bundle) satisfies the representation-theoretic Howe condition. In the last section we illustrate the results with some simple but instructive examples.

2 Properties of Commuting Hamiltonian Actions

The setting throughout this article is a smooth connected symplectic manifold (M, ω) on which two smooth actions of the Lie groups G_1 and G_2 are given.

Assume these actions commute and are hamiltonian such that each action admits an equivariant moment map $\Phi_i : M \rightarrow \mathfrak{g}_i^*$ ($i = 1, 2$), i.e., they satisfy $d\Phi_i^\xi = \omega(\tau_i(\xi), \cdot)$ and $\Phi_i(g \cdot z) = \text{Ad}^*(g)\Phi_i(z) \forall z \in M, g \in G_i$. Here, for any $\xi \in \mathfrak{g}_i$, we denote by $\tau_i(\xi)$ the fundamental vector field on M associated to ξ and by Φ_i^ξ the component of the i -moment map in direction ξ . We first notice the following easy lemma.

Lemma 2.1. *Let (M, ω) be a symplectic manifold. Let the Lie groups G_1 and G_2 act symplectically on M , admitting equivariant moment maps Φ_1 and Φ_2 . Assume these actions commute. Then the product group $G_1 \times G_2$ acts symplectically on M with an equivariant moment map, and notably for any $\xi \in \mathfrak{g}_1$ and any $\eta \in \mathfrak{g}_2$ the Poisson bracket of the moment components vanishes, i.e.,*

$$\{\Phi_1^\xi, \Phi_2^\eta\} = 0.$$

Proof. Obvious. □

In other words, the moment map of the first action is constant along the connected components of the orbits of the second action, and vice versa. For G_1 and G_2 (or, at least, their orbits) connected, this can be rephrased as

$$\Phi_i^* C^\infty(\mathfrak{g}_i^*) \subseteq C^\infty(M)^{G_j} \quad \text{for } i \neq j.$$

Here we observe the inclusion of the collective functions for the first action in the set of invariants for the second action, and vice versa. Let us now observe the following useful -and most certainly folkloristic- result.

Lemma 2.2. *Let G be a connected Lie group and (M, ω) a symplectic manifold. If G acts symplectically with an equivariant moment map Φ on M , then*

$$Z_{C^\infty(M)}(\Phi^* C^\infty(\mathfrak{g}^*)) = C^\infty(M)^G.$$

Proof. Note that for $f \in C^\infty(M)$ and $\xi \in \mathfrak{g}$, one has that $\xi^M(f) = \{f, \Phi^\xi\}$ (where $\xi^M = \tau(\xi)$ denotes the fundamental vector field on M associated to $\xi \in \mathfrak{g}$), and, furthermore, if $z \in M$, one has $\{f, \Phi^\xi\}(z) = -\langle \xi, T_z \Phi(X_{f|z}) \rangle$ where $\langle \cdot, \cdot \rangle$ is the natural pairing between \mathfrak{g} and \mathfrak{g}^* and $\omega(X_f, \cdot) = df$, i.e., X_f is the hamiltonian vector field associated to f .

Given now $f \in C^\infty(M)^G$ one has that for all $z \in M$, $T_z \Phi(X_{f|z}) = 0$. If $h \in C^\infty(\mathfrak{g}^*)$ and $z \in M$, we have

$$\{h \circ \Phi, f\}(z) = d(h \circ \Phi)_z(X_{f|z}) = dh_{\Phi(z)}(T_z \Phi(X_{f|z})) = 0.$$

On the other hand, if $f \in Z_{C^\infty(M)}(\Phi^* C^\infty(\mathfrak{g}^*))$ and $\xi \in \mathfrak{g}$, then $\xi^M(f) = \{f, \Phi^\xi\} = 0$, and thus by the connectedness of G , $f \in C^\infty(M)^G$. □

Remark 2.3. More subtle relations between centralizers of pullbacks via Φ and invariant functions are shown in [KL97] in the case that G is compact (often extending, in fact, to proper actions).

With the two preceding lemmata we immediately get that

$$\Phi_i^* C^\infty(\mathfrak{g}_i^*) \subseteq Z_{C^\infty(M)}(\Phi_j^* C^\infty(\mathfrak{g}_j^*)) \quad \text{for } i \neq j.$$

Equality of the respective sets is obviously the classical analogue of (**) upon interpreting $\Phi^* C^\infty(\mathfrak{g}^*)$ as the “classical collective observables (w. r. t. to a hamiltonian G -action on M)” and $\varrho(\mathcal{U}\mathfrak{g})$ as the “quantum collective observables (w. r. t. to a linear G -representation)”. Therefore, it is natural to study the following situation.

Definition 2.4. We say that two commuting hamiltonian actions satisfy the *symplectic Howe condition* or form a *symplectic Howe pair* if

$$Z_{C^\infty(M)}(\Phi_i^* C^\infty(\mathfrak{g}_i^*)) = \Phi_j^* C^\infty(\mathfrak{g}_j^*) \quad \text{for } i \neq j$$

is satisfied.

Remark 2.5. The above condition is clearly close to the notions of dual pairs presented in Ch. 11 of [OR04]. However, we do not require the moment maps to be submersions nor surjections when dealing with the symplectic Howe condition.

Assuming further the properness of both actions, the symplectic Howe condition has the following important consequence for the orbit structure of these commuting actions.

Proposition 2.6. *Let commuting hamiltonian proper actions of the connected Lie groups G_1 and G_2 with equivariant moment maps Φ_1 and Φ_2 be given on the symplectic manifold (M, ω) . Then the symplectic Howe condition*

$$Z_{C^\infty(M)}(\Phi_i^* C^\infty(\mathfrak{g}_i^*)) = \Phi_j^* C^\infty(\mathfrak{g}_j^*) \quad \text{for } i \neq j$$

implies that

$$\forall z \in M \text{ holds } \Phi_i^{-1}(\Phi_i(z)) = G_j \cdot z \quad \text{for } i \neq j,$$

i. e., the levels sets of the moment maps of one action are the orbits of the other one.

Proof. Let $i, j \in \{1, 2\}$ such that $i + j = 3$. Fix $z \in M$ and denote $\Phi_i^{-1}(\Phi_i(z)) = N_z$. By Lemma 2.1 and the connectedness of G_j , the G_j -action on M preserves the subset $N_z \subseteq M$.

Assume that $N_z \neq G_j \cdot z$. Then there exists $z' \in N_z$ such that $G_j \cdot z' \cap G_j \cdot z = \emptyset$. Since the G_j -action is proper, there exists $f \in C^\infty(M)^{G_j}$ such that $f(z') \neq f(z)$ (separation of orbits by invariant functions follows easily from the slice theorem for proper actions).

By Lemma 2.2 and the symplectic Howe condition, we have

$$C^\infty(M)^{G_j} = Z_{C^\infty(M)}(\Phi_j^* C^\infty(\mathfrak{g}_j^*)) = \Phi_i^* C^\infty(\mathfrak{g}_i^*),$$

i.e., there is a smooth function $h : \mathfrak{g}_i^* \rightarrow \mathbb{R}$ such that $f = h \circ \Phi_i$. Thus we arrive at $f(z') = h(\Phi_i(z')) = h(\Phi_i(z)) = f(z)$, contradicting our assumption. Consequently, $\Phi_i^{-1}(\Phi_i(z)) = G_j \cdot z$. \square

Knowing the level sets of the moment maps as precisely as above permits to relate the stabilizers of the actions on M and on the moment images $\Phi_i(M)$.

Lemma 2.7. *Let G_1 and G_2 be Lie groups and (M, ω) be a symplectic manifold. Let hamiltonian actions of both groups on M be given which commute, and denote the equivariant moment maps by $\Phi_i : M \rightarrow \mathfrak{g}_i^*$. Let $G_{12,z} = \{(g, g') \in G_1 \times G_2 \mid (g, g') \cdot z = z\}$ be the stabilizer of a point $z \in M$ under the simultaneous action of both groups, write $H_{1,z}$ and $H_{2,z}$ for the projections of $G_{12,z}$ to the groups G_1 and G_2 .*

Then

- (i) $H_{1,z}$ is contained in $G_{1,\Phi_1(z)}$, the stabilizer of the image of z under the coadjoint action (i.e., $G_{1,z} \subseteq H_{1,z} \subseteq G_{1,\Phi_1(z)}$), and
- (ii) if, furthermore, the level sets of Φ_1 are actually G_2 -orbits, then $H_{1,z} = G_{1,\Phi_1(z)}$.

The same statements hold for the indices 1 and 2 interchanged.

Proof. (i) Take $g \in H_{1,z}$, i.e., there exists $g' \in G_2$ such that $(g, g') \cdot z = z$. From this one obtains

$$\Phi_1(z) = \Phi_1((g, g') \cdot z) = g \cdot \Phi_1((e, g') \cdot z) = g \cdot \Phi_1(z),$$

where G_1 -equivariance and G_2 -invariance of Φ_1 have been used.

- (ii) Now take $g \in G_{1,\Phi_1(z)}$, then $\Phi_1(g \cdot z) = \Phi_1(z)$, i.e., both z and $g \cdot z$ lie in a level set of Φ_1 , which is by assumption a G_2 -orbit. Thus there exists $g' \in G_2$ such that $(g, g') \cdot z = z$, which was to be shown.

Of course, interchanging both actions does not alter the proof. \square

The particular orbit structure that we have described yields now a correspondence between the orbits in the images of the moment maps Φ_1 and Φ_2 .

Theorem 2.8. *Let commuting hamiltonian proper actions of the connected Lie groups G_1 and G_2 with equivariant moment maps Φ_1 and Φ_2 be given on the symplectic manifold (M, ω) .*

If the symplectic Howe condition is satisfied, then:

(i) There is a bijection $\Lambda : \Phi_1(M)/G_1 \rightarrow \Phi_2(M)/G_2$, given by

$$\Lambda(\mathcal{O}_{\alpha_1}) = \Phi_2(\Phi_1^{-1}(\mathcal{O}_{\alpha_1})),$$

where $\alpha_1 \in \Phi_1(M)$, and $\mathcal{O}_{\alpha_1} = \text{Ad}^*(G_1)\alpha_1$ is seen as an element of $\Phi_1(M)/G_1$. To each pair $(\mathcal{O}_{\alpha_1}, \Lambda(\mathcal{O}_{\alpha_1}))$ belongs a unique orbit $(G_1 \times G_2) \cdot z$ in M given by $\Phi_1^{-1}(\mathcal{O}_{\alpha_1}) = \Phi_2^{-1}(\mathcal{O}_{\alpha_2})$. We say that orbits $(G_1 \times G_2) \cdot z$, \mathcal{O}_{α_1} and \mathcal{O}_{α_2} are in correspondence if $\mathcal{O}_{\alpha_2} = \Lambda(\mathcal{O}_{\alpha_1})$ and $(G_1 \times G_2) \cdot z = \Phi_1^{-1}(\mathcal{O}_{\alpha_1}) = \Phi_2^{-1}(\mathcal{O}_{\alpha_2})$.

(ii) If Φ_1 and Φ_2 are open maps, then Λ is a homeomorphism. The same conclusion holds if the groups G_1 and G_2 both are compact and Φ_1 and Φ_2 both are closed maps. Finally, the same conclusion holds if for $i + j = 3$, G_i is compact and Φ_i is closed, and Φ_j is open.

(iii) Write $M_{\alpha_1} = \Phi_1^{-1}(\alpha_1)/G_{1,\alpha_1}$ and $M_{\alpha_2} = \Phi_2^{-1}(\alpha_2)/G_{2,\alpha_2}$ for the respective point reduced spaces at $\alpha_1 \in \mathfrak{g}_1^*$ and $\alpha_2 \in \mathfrak{g}_2^*$. These spaces can be described as coadjoint orbits of the other action, i. e., there are, resp., G_2 - and G_1 -equivariant symplectomorphisms

$$M_{\alpha_1} \rightarrow \Lambda(\mathcal{O}_{\alpha_1}) \quad \text{and} \quad M_{\alpha_2} \rightarrow \Lambda^{-1}(\mathcal{O}_{\alpha_2}).$$

(iv) The symplectic reduced space $M_{(\alpha_1, \alpha_2)}$ for the joint action of $G_1 \times G_2$ is either a point (if \mathcal{O}_{α_1} and \mathcal{O}_{α_2} are in correspondence) or empty otherwise.

Proof. In order to simplify indices, some statements will only be proved for one action if the other case is analogous.

(i) By Prop. 2.6, we know that for any $z \in M$, the level sets of both moment maps are orbits: $\Phi_i^{-1}(\Phi_i(z)) = G_j \cdot z$ ($i + j = 3$). This implies that the preimage of any coadjoint orbit in either moment image $\Phi_i(M)$ ($i = 1, 2$) is exactly one orbit of the joint action of $G_1 \times G_2$ on M , i. e.,

$$\Phi_i^{-1}(\text{Ad}^*(G_i)\Phi_i(z)) = (G_1 \times G_2) \cdot z,$$

from which it follows that Λ is well-defined and bijective.

(ii) Note that all maps in the diagram

$$\Phi_1(M)/G_1 \xleftarrow{\pi_1} \Phi_1(M) \xleftarrow{\Phi_1} M \xrightarrow{\Phi_2} \Phi_2(M) \xrightarrow{\pi_2} \Phi_2(M)/G_2$$

are continuous, and π_1 and π_2 are always open as well.

Let us denote $\Lambda : \Phi_1(M)/G_1 \rightarrow \Phi_2(M)/G_2$ by Λ_{12} and its set-theoretic inverse by Λ_{21} . If Φ_1 is open, then for U open in $\Phi_2(M)/G_2$ we have that $\Lambda_{21}(U) = (\pi_1 \circ \Phi_1)(\Phi_2^{-1}(\pi_2^{-1}(U)))$ is an open set, i. e., Λ_{12} is continuous. Similarly, if G_1 is compact and Φ_1 is closed, we have for A closed in $\Phi_2(M)/G_2$ that $\Lambda_{21}(A)$ is closed as well, i. e., again Λ_{12} is continuous.

The conclusions of (ii) follow easily.

- (iii) Let $z \in M$ and $\alpha_2 = \Phi_2(z)$ its value under the moment map of the second action. Consider the restricted map $\Phi_{2|G_2 \cdot z} : G_2 \cdot z \rightarrow \mathcal{O}_{\alpha_2}$, and recall that $G_2 \cdot z = \Phi_1^{-1}(\alpha_1)$ for $\alpha_1 = \Phi_1(z)$. Recall from Lemma 2.7 that $G_{1,\alpha_1} = \{h \in G_1 \mid \exists g \in G_2 : (h, g) \cdot z = z\}$. This group acts on $G_2 \cdot z$ and one has $M_{\alpha_1} = G_2 \cdot z / G_{1,\alpha_1}$. Thus $\Phi_{2|G_2 \cdot z}$ induces

$$\tilde{\Phi}_2 : M_{\alpha_1} \rightarrow \mathcal{O}_{\alpha_2},$$

which inherits from Φ_2 smoothness and G_2 -equivariance. It is clearly surjective and we now show that it is injective: Take $\alpha \in \mathcal{O}_{\alpha_2}$, $\tilde{z}_1, \tilde{z}_2 \in G_2 \cdot z / G_{1,\alpha_1}$ so that $\alpha = \tilde{\Phi}_2(\tilde{z}_1) = \tilde{\Phi}_2(\tilde{z}_2)$. Now fixing preimages $z_1, z_2 \in G_2 \cdot z$ of \tilde{z}_1, \tilde{z}_2 , they have the property $\Phi_2(z_1) = \Phi_2(z_2)$, and for some $h \in G_1$, $z_2 = h \cdot z_1$ holds because the level sets of Φ_2 are G_1 -orbits. However, $z_2 = g \cdot z_1$ for some $g \in G_2$, which implies by part (ii) of Lemma 2.7 that $h \in G_{1,\alpha_1}$. Therefore, $\tilde{z}_1 = \tilde{z}_2$, and $\tilde{\Phi}_2$ is a bijection. Applying Sard's Theorem, one notices that a smooth equivariant bijection between finite-dimensional homogeneous spaces has a smooth inverse. Thus $\tilde{\Phi}_2$ is a G_2 -equivariant diffeomorphism and it remains to show that $\tilde{\Phi}_2$ is a symplectomorphism. Denote by $i_{G_2 \cdot z} : G_2 \cdot z \rightarrow M$ the inclusion of $G_2 \cdot z = \Phi_1^{-1}(\alpha_1)$ into the ambient manifold. We observe that the equivariance properties of Φ_2 imply $i_{G_2 \cdot z}^* \omega = (\Phi_{2|G_2 \cdot z})^* \omega^{\mathcal{O}_{\alpha_2}}$, where $\omega^{\mathcal{O}_{\alpha_2}}$ is the KKS symplectic form (see, e. g., [OR04], Thm. 4.5.31). But the symplectic form $\omega^{M_{\alpha_1}}$ on M_{α_1} is defined such that it also pulls back to $i_{G_2 \cdot z}^* \omega = p^* \omega^{M_{\alpha_1}}$, via the quotient map $p : G_2 \cdot z \rightarrow G_2 \cdot z / G_{1,\alpha_1}$. Consequently we have $p^* \omega^{M_{\alpha_1}} = (\Phi_{2|G_2 \cdot z})^* \omega^{\mathcal{O}_{\alpha_2}} = p^* (\tilde{\Phi}_2^* \omega^{\mathcal{O}_{\alpha_2}})$. Since p is a surjective submersion, the coincidence of the pullbacks (to $G_2 \cdot z$) implies that $\omega^{M_{\alpha_1}} = \tilde{\Phi}_2^* \omega^{\mathcal{O}_{\alpha_2}}$, which was to be shown.

- (iv) Let $\Phi = \Phi_1 \oplus \Phi_2$. Take $\alpha_1 \in \mathfrak{g}_1^*$ and $\alpha_2 \in \mathfrak{g}_2^*$. Then

$$\Phi^{-1}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2})$$

is empty if $\mathcal{O}_{\alpha_2} \neq \Lambda(\mathcal{O}_{\alpha_1})$. Otherwise, for any $z \in M$ such that $\Phi_1(z) \in \mathcal{O}_{\alpha_1}$ and $\Phi_2(z) \in \mathcal{O}_{\alpha_2}$, holds

$$\Phi^{-1}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2}) / (G_1 \times G_2) \cong (G_1 \times G_2) \cdot z / (G_1 \times G_2).$$

This quotient is, of course, a point.

□

Remark 2.9. (1) Statement (i) is a special case of the singular symplectic leaf correspondence of Thm. 11.4.4 in [OR04], but here obtained from the symplectic Howe condition on the pair of groups actions, in a spirit close to the non-singular correspondence of Thm. 11.1.9 of the same reference.

- (2) Take any $z \in M$ and $\alpha_1 = \Phi_1(z)$. Then the global ineffectivity of the G_{1,α_1} -action on the Φ_1 -level containing z is $I_{1,\alpha_1} = \{g_1 \in G_{1,\alpha_1} \mid g_1 \cdot z' =$

$z' \forall z' \in \Phi_1^{-1}(\alpha_1)\}$, i. e., the intersection of the stabilizers of all points in the level set. Here, $\Phi_1^{-1}(\alpha_1) = G_2 \cdot z$, hence $G_{1,z'} = G_{1,z} \subseteq I_{1,\alpha_1}$ for all $z' \in \Phi_1^{-1}(\alpha_1)$, so $I_{1,\alpha_1} = G_{1,z}$. Therefore, the proper action of G_{1,α_1} on $\Phi_1^{-1}(\alpha_1)$ factorizes over a free and proper action of $G_{1,\alpha_1}/I_{1,\alpha_1}$, and thus the quotient is a smooth manifold.

- (3) Part (iv) of the preceding theorem implies that the $(G_1 \times G_2)$ -action on M is multiplicity-free, i. e., $C^\infty(M)^{G_1 \times G_2}$ is a commutative Poisson algebra (compare [GS84] and [HW90]). This can also be directly deduced from the definition of a symplectic Howe pair and Lemma 2.2.

3 Prequantization of Symplectic Howe Pairs

We are now going to show that the orbit correspondence which was constructed above behaves well under geometric prequantization. The first step is to show that the integrality of coadjoint orbits is preserved under the correspondence. To achieve this, we make use of having proved that the reduced spaces coincide in our setting with coadjoint orbits. Let us recall the following fact (Thm. 3.2 of [GS82]).

Theorem 3.1. *Let G be a compact connected Lie group and (M, ω) be a prequantizable symplectic manifold with hamiltonian G -action, equivariant moment map Φ and line bundle $L(M)$ equipped with a connection ∇ on $L(M)$ whose curvature coincides with ω . If the G -action on the level set $\Phi^{-1}(0)$ is free, then the reduced space $M_0 = \Phi^{-1}(0)/G$ is a manifold and there exists a unique line bundle $L(M_0)$ over M_0 such that*

$$\pi^*L(M_0) = i^*L(M) \text{ and } \pi^*\nabla_0 = i^*\nabla,$$

where $\pi : \Phi^{-1}(0) \rightarrow \Phi^{-1}(0)/G$ is the quotient map and $i : \Phi^{-1}(0) \rightarrow M$ the inclusion.

Notation. If (M, ω) is a prequantizable symplectic manifold, i. e., ω is an integral form, we denote by $L(M, \omega)$ a corresponding prequantum line bundle, i. e., a complex line bundle with first de Rham Chern class equal to $[\omega]$. This line bundle is unique if there exist no torsion line bundles on M . We may omit ω if there is no ambiguity about the symplectic form. For a coadjoint orbit \mathcal{O}_α , we will write $L_\alpha = L(\mathcal{O}_\alpha)$, the KKS symplectic form $\omega^{\mathcal{O}_\alpha}$ being understood.

By the shifting trick, any reduced space may be regarded as a reduced space at 0. More precisely, one has the following well-known result (see, e. g., Thm. 6.5.2 in [OR04]).

Theorem 3.2. *Let G be a compact connected Lie group and (M, ω) be a prequantizable symplectic manifold with hamiltonian G -action and equivariant moment map Φ . If G_α acts freely on $\Phi^{-1}(\alpha)$, then the reduced space at $\alpha \in \mathfrak{g}^*$, $\Phi^{-1}(\alpha)/G_\alpha$, is symplectomorphic to the reduction of $M \times \mathcal{O}_\alpha^-$ at 0, where \mathcal{O}_α^-*

denotes the coadjoint orbit through α with the KKS symplectic form multiplied by -1 .

Returning to the setting of Theorem 2.8, we conclude that any reduced space M_{α_1} admits a line bundle $L(M_{\alpha_1})$ if the coadjoint orbit \mathcal{O}_{α_1} does. Recall from Remark 2.9(2) that the quotient of $\Phi_1^{-1}(\alpha_1)$ by G_{1,α_1} is also given as the quotient by the free $(G_{1,\alpha_1}/I_{1,\alpha_1})$ -action on $\Phi_1^{-1}(\alpha_1)$ and hence M_{α_1} is smooth. Therefore, we can adapt Theorem 3.1 to this quotient. Using the symplectomorphism $\tilde{\Phi}_2 : M_{\alpha_1} \rightarrow \mathcal{O}_{\alpha_2}$, we obtain the bundle $(\tilde{\Phi}_2^{-1})^* L(M_{\alpha_1})$ over \mathcal{O}_{α_2} . As coadjoint orbits of compact connected Lie groups are simply connected, there are no non-trivial flat vector bundles, hence no torsion line bundles over them. Therefore, for any α_1 such that $(\mathcal{O}_{\alpha_1}, \omega^{\mathcal{O}_{\alpha_1}})$ is an integral symplectic manifold, the bundle $(\tilde{\Phi}_2^{-1})^* L(M_{\alpha_1})$ is the unique prequantum line bundle over \mathcal{O}_{α_2} . Thus we have proved:

Proposition 3.3. *Let \mathcal{O}_{α_1} and \mathcal{O}_{α_2} be two coadjoint orbits in correspondence as in Thm. 2.8(i) and assume furthermore that G_1 and G_2 are compact. Then $(\mathcal{O}_{\alpha_1}, \omega^{\mathcal{O}_{\alpha_1}})$ is an integral symplectic manifold if and only if $(\mathcal{O}_{\alpha_2}, \omega^{\mathcal{O}_{\alpha_2}})$ is an integral symplectic manifold, too.*

Remark 3.4. Let us recall that for G a compact connected semisimple Lie group and $\alpha \in \mathfrak{g}^*$, \mathcal{O}_α is an integral symplectic manifold if and only if there exists a character $\chi_\alpha : G_\alpha \rightarrow U(1)$ such that $(\chi_\alpha)_{*e} = 2\pi i \alpha$. Thus one defines:

Definition 3.5. (1) Let G be a compact connected Lie group and $\alpha \in \mathfrak{g}^*$. We call α *integral* if there exists $\chi_\alpha : G_\alpha \rightarrow U(1)$ such that $(\chi_\alpha)_{*e} = 2\pi i \alpha$.
(2) If α is integral, we call \mathcal{O}_α an *integral orbit*.

Remark 3.6. Of course, there is a natural bijection between the set of integral orbits and the set of dominant weights of G (w. r. t. a fixed Weyl chamber \mathfrak{t}^+).

We observe that integral coadjoint orbits are integral symplectic manifolds but the converse statement is not true in general because of the possible presence of a positive-dimensional centre. Accordingly, we need to refine Proposition 3.3.

Let us from now on assume that the $(G_1 \times G_2)$ -action on M comes with a fixed “linearization on L ”, i. e., there is given a fibrewise linear $(G_1 \times G_2)$ -action on $L = L(M, \omega)$ covering the $(G_1 \times G_2)$ -action on M . This is no restriction if one accepts to replace G_1 and G_2 by finite coverings (see [Dui96], Prop. 15.4).

Theorem 3.7. *Let \mathcal{O}_{α_1} and \mathcal{O}_{α_2} be two coadjoint orbits in correspondence as in Thm. 2.8(i) and assume furthermore that G_1 and G_2 are compact. Then \mathcal{O}_{α_1} is integral if and only if \mathcal{O}_{α_2} is integral.*

Proof. Let $\alpha \in \mathfrak{g}_1^*$ be integral. Since the G_2 -actions on M and L commute with the G_1 -actions, the G_2 -action on L descends canonically to a G_2 -action on the line bundle $L(M_{\alpha_1})$ covering the natural G_2 -action on M_{α_1} .

Using the G_2 -equivariant symplectomorphism $\tilde{\Phi}_2 : (M_{\alpha_1}, \omega^{M_{\alpha_1}}) \rightarrow (\mathcal{O}_{\alpha_2}, \omega^{\mathcal{O}_{\alpha_2}})$ from above, we have the bundle $(\tilde{\Phi}_2^{-1})^* L(M_{\alpha_1}) = L(\mathcal{O}_{\alpha_2})$ together with a G_2 -invariant connection ∇ given by pulling back the G_2 -invariant connection induced on $L(M_{\alpha_1})$ by a $(G_1 \times G_2)$ -invariant connection on $L \rightarrow M$. (See [GS82] for the construction of the connection on $L(M_{\alpha_1})$.) We thus arrive at the following commuting diagram (where p denotes the bundle projection and $L(\mathcal{O}_{\alpha_2})_{\alpha_2} = p^{-1}(\alpha_2)$):

$$\begin{array}{ccc} G_2 \times L(\mathcal{O}_{\alpha_2}) & \longrightarrow & L(\mathcal{O}_{\alpha_2}) \\ \text{id} \times p \downarrow & & \downarrow p \\ G_2 \times \mathcal{O}_{\alpha_2} & \longrightarrow & \mathcal{O}_{\alpha_2}, \end{array}$$

yielding a homomorphism $\chi : G_{2,\alpha_2} \rightarrow U(L(\mathcal{O}_{\alpha_2})_{\alpha_2}) = U(1)$.

It remains to show that $\chi = \chi_{\alpha_2}$, i.e., $\chi_{*e} = 2\pi i \alpha_2$. Since the G_2 -action on $L(\mathcal{O}_{\alpha_2})$ and the connection ∇ come from the reduced bundle with connection $L(M_{\alpha_1})$ and this comes in turn from a connection on $L \rightarrow M$, we have the usual ‘‘Kostant formula’’ for the fundamental vector fields of the G_2 -action on $L(\mathcal{O}_{\alpha_2})$. (Compare [Kos70], Thm. 4.5.1.) More precisely, given $\xi \in \mathfrak{g}_2$, $\xi^{L(\mathcal{O}_{\alpha_2})} = \xi^{\mathcal{O}_{\alpha_2}} + p^*(\langle \Phi_2^{\mathcal{O}_{\alpha_2}}, \xi \rangle)(2\pi i)^{L(\mathcal{O}_{\alpha_2})}$, where ξ^N denotes the fundamental vector field associated to ξ on a G_2 -manifold N , \tilde{X} denotes the ∇ -horizontal lift of a vector field X on \mathcal{O}_{α_2} to $L(\mathcal{O}_{\alpha_2})$, $2\pi i \in i\mathbb{R} \cong \mathfrak{u}(1)$ also has a fundamental vector field on the bundle by the canonical $U(1)$ -action on it, and $\langle \Phi_2^{\mathcal{O}_{\alpha_2}}, \xi \rangle$ is the ξ -component of the G_2 -moment map on \mathcal{O}_{α_2} .

For $\xi \in \mathfrak{g}_{2,\alpha_2}$, the field $\xi^{L(\mathcal{O}_{\alpha_2})}$ is now tangent to the p -fibre $L(\mathcal{O}_{\alpha_2})_{\alpha_2}$ over α_2 and equals there $(2\pi i)^{L(\mathcal{O}_{\alpha_2})} \langle \alpha_2, \xi \rangle$. Thus $\chi_{*e} : G_{2,\alpha_2} \rightarrow U(1)$ is equal to $\frac{d}{dt}\big|_0 e^{2\pi i \langle \alpha_2, \xi \rangle t} = 2\pi i \langle \alpha_2, \xi \rangle$. \square

4 Geometric Quantization of Symplectic Howe Pairs

In this section we show that the symplectic Howe condition for the actions on M implies the representation-theoretic Howe condition for the linear representation on the geometric quantization of (M, ω) in the Kähler case. Thus we assume throughout this section that (M, ω) is Kähler, i.e., there is given a complex structure J such that the associated riemannian metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is positive definite and hermitean. Furthermore we assume that the acting groups are compact and connected and that the moment maps are admissible in the sense of [Sja95] (p. 109).

Definition 4.1. Let G be a compact connected Lie group acting holomorphically on a Kähler manifold M with equivariant moment map Φ . Choose a G -invariant inner product on \mathfrak{g} with corresponding norm and define the function $\mu = \|\Phi\|^2$.

Let F_t be the gradient flow of $-\mu$. A moment map is called *admissible* if for every $z \in M$, the path of steepest descent $F_t(z)$ through z ($t \geq 0$) is contained in a compact set.

Remark 4.2. Examples of admissible moment maps are all proper moment maps and the moment map of the natural linear $U(n)$ -action on \mathbb{C}^n .

One then has the following result (Thm. 2.20 of [Sja95]).

Theorem 4.3. *Let G be a compact connected Lie group acting holomorphically on a Kähler manifold M with admissible equivariant moment map Φ . Suppose this action extends to an action of the complexification $G^\mathbb{C}$ of G . Then for every dominant weight α of G , the space of holomorphic sections $\Gamma_{\text{hol}}(M_\alpha, L_\alpha)$ of the prequantum line bundle L_α over the symplectic reduced space M_α is naturally isomorphic to $\text{Hom}_G(V_\alpha, \Gamma_{\text{hol}}(M, L))$, the space of intertwining operators from the irreducible representation V_α with highest weight α to the quantization $\Gamma_{\text{hol}}(M, L)$ of M .*

- Remark 4.4.** (1) Of course, *dominance* of a weight (an analytically integral element of \mathfrak{t}^*) is defined w. r. t. the choice of a Weyl chamber. It is introduced in the statement of the preceding theorem only to avoid redundancy since one has $M_{\text{Ad}^*(g)\alpha} \cong M_\alpha$ and $V_{\text{Ad}^*(g)\alpha} \cong V_\alpha$. Thus one may write here *integral point of \mathfrak{g}^** in the sense of Def. 3.5 as well.
- (2) The reduced spaces M_α inherit, also in the singular case, “sufficient” structure in order to define holomorphic sections of $L_\alpha \rightarrow M_\alpha$ (compare [Sja95]). By Theorem 2.8 and Remark 2.9(2) the reduced spaces occurring in our “symplectic Howe setting” are always smooth. Thus, in fact, we do not need the results of Sjamaar in its full generality.
- (3) If α , interpreted as an element of \mathfrak{g}^* , is not in the moment image $\Phi(M)$, then $\Gamma_{\text{hol}}(M_\alpha, L_\alpha)$ is to be interpreted as $\{0\}$. Theorem 4.3 then says that V_α does not occur in the G -decomposition of $\Gamma_{\text{hol}}(M, L)$.

Take now the simultaneous $G_1 \times G_2$ -action introduced in the previous section. By the Borel-Weil theorem, the geometric quantizations, $V_{\alpha_i} \cong \Gamma_{\text{hol}}(\mathcal{O}_{\alpha_i}, L_{\alpha_i})$ with α_i integral, realize the irreducible representations of G_i , thus $\Gamma_{\text{hol}}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2}, L_{\alpha_1} \boxtimes L_{\alpha_2})$ the irreducibles for $G_1 \times G_2$. Here $L_{\alpha_1} \boxtimes L_{\alpha_2} \rightarrow M_1 \times M_2$ is given as $p_1^*(L_{\alpha_1}) \otimes p_2^*(L_{\alpha_2})$ with $p_j : M_1 \times M_2 \rightarrow M_j$ denoting the j -th projection for $j = 1, 2$. Now,

$$\begin{aligned} \text{Hom}_{G_1 \times G_2}(\Gamma_{\text{hol}}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2}, L_{\alpha_1} \boxtimes L_{\alpha_2}), \Gamma_{\text{hol}}(M, L)) \\ \cong \Gamma_{\text{hol}}(\Phi^{-1}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2})/(G_1 \times G_2), L_{(\alpha_1, \alpha_2)}). \end{aligned}$$

By Thm. 2.8(iv), the reduced space may be empty (if the coadjoint orbits are not in correspondence), hence the space of sections $\Gamma_{\text{hol}}(\Phi^{-1}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2})/(G_1 \times G_2), L_{(\alpha_1, \alpha_2)})$ is trivial; otherwise the reduced space is a point and the space

of sections is simply \mathbb{C} . So, one concludes for the multiplicities of $G_1 \times G_2$ -representations in the quantization of M :

$$\dim \text{Hom}_{G_1 \times G_2} (\Gamma_{\text{hol}}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2}, L_{\alpha_1} \boxtimes L_{\alpha_2}), \Gamma_{\text{hol}}(M, L)) \leq 1,$$

the equal sign being true if and only if $\mathcal{O}_{\alpha_2} = \Lambda(\mathcal{O}_{\alpha_1})$.

Interpreting the line bundles $L_{\alpha_1}, L_{\alpha_2}$ and $L_{\alpha_1} \boxtimes L_{\alpha_2}$ as sheaves and their holomorphic sections as their zeroth cohomology group, one concludes from a standard Künneth formula (compare [SW59] and [Kau67]) that

$$\Gamma_{\text{hol}}(\mathcal{O}_{\alpha_1} \times \mathcal{O}_{\alpha_2}, L_{\alpha_1} \boxtimes L_{\alpha_2}) \cong \Gamma_{\text{hol}}(\mathcal{O}_{\alpha_1}, L_{\alpha_1}) \otimes \Gamma_{\text{hol}}(\mathcal{O}_{\alpha_2}, L_{\alpha_2}).$$

Recall that by 2.8(iii), the coadjoint G_2 -orbit corresponding via Λ to $\mathcal{O}_{\alpha_1} \in \Phi_1(M)/G_1$ is symplectomorphic to the orbit reduced space of \mathcal{O}_{α_1} , i.e.,

$$\Lambda(\mathcal{O}_{\alpha_1}) \cong \Phi_1^{-1}(\mathcal{O}_{\alpha_1})/G_1 = M_{\alpha_1}.$$

This implies that the multiplicity space of one action is an irreducible representation of the other action, i.e., for $\mathcal{O}_{\alpha_2} = \Lambda(\mathcal{O}_{\alpha_1})$, one has G_2 -equivariantly

$$\text{Hom}_{G_1}(V_{\alpha_1}, \Gamma_{\text{hol}}(M, L)) \cong \Gamma_{\text{hol}}(M_{\alpha_1}, L(M_{\alpha_1})) \cong \Gamma_{\text{hol}}(\mathcal{O}_{\alpha_2}, L_{\alpha_2}).$$

Let us identify in the sequel (for $k = 1, 2$) \widehat{G}_k , the set of equivalence classes of irreducible complex representations of finite dimension of G_k , with $(\mathfrak{t}_k)_{\mathbb{Z}}^+$ the integral points in a fixed Weyl chamber \mathfrak{t}_k^+ in the dual \mathfrak{t}_k^* of a maximal abelian subalgebra $\mathfrak{t}_k \subseteq \mathfrak{g}_k$.

The preceding statements can be summarized as follows.

Theorem 4.5. *Let G_1 and G_2 be compact connected Lie groups acting by holomorphic transformations on a Kähler manifold M such that the actions extend to actions of the respective complexified groups. Suppose that the actions of G_1 and G_2 commute and are hamiltonian with admissible equivariant moment maps Φ_1 and Φ_2 . Denote by L the prequantum line bundle over M .*

Assume the symplectic Howe condition to be satisfied. Then:

$$\Gamma_{\text{hol}}(M, L) \cong \bigoplus_{\alpha_1 \in \Phi_1(M) \cap \widehat{G}_1} \Gamma_{\text{hol}}(\mathcal{O}_{\alpha_1}, L_{\alpha_1}) \otimes \Gamma_{\text{hol}}(\mathcal{O}_{\alpha_2}, L_{\alpha_2}),$$

where Λ is the orbit correspondence map and $\mathcal{O}_{\alpha_2} = \Lambda(\mathcal{O}_{\alpha_1})$.

Remark 4.6. (1) In the preceding theorem the symbol $\widehat{\bigoplus}$ of course signifies the completion of the algebraic direct sum in the Fréchet topology of the space of holomorphic sections of L .

(2) Since Λ , viewed as a map from $\Phi_1(M) \cap \widehat{G}_1$ to \widehat{G}_2 , is injective the representation $\Gamma_{\text{hol}}(M, L)$ of the pair (G_1, G_2) satisfies the representation-theoretic Howe condition $(*)$ recalled in the introduction.

- (3) Let (M, ω) be a compact complex manifold together with an integral Kähler form and $L \rightarrow M$ a prequantizing holomorphic line bundle. Let furthermore the connected compact Lie groups G_1 and G_2 act by holomorphic transformations and in a hamiltonian fashion on M . If this pair of actions satisfies the symplectic Howe condition, the preceding theorem applies, i.e., their induced linear representation on the “holomorphic quantization” $\Gamma_{\text{hol}}(M, L)$ is equipped with a Howe duality.

5 Examples

In this section we give some simple but instructive examples of pairs of symplectic actions fulfilling the symplectic Howe condition. In each case, we explicitly give the orbit correspondence map and the Howe duality map associated to the linear representation of the pair of groups on the geometric quantization space.

5.1 $(U(n), U(m))$ on $\text{Mat}(n \times m; \mathbb{C})$

Let $M = \text{Mat}(n \times m; \mathbb{C})$ be the $n \times m$ matrices with complex entries equipped with the symplectic structure $\omega(A, B) = \text{Im tr}(\bar{A}^T B)$ and the natural actions of $G_1 = U(n)$ and $G_2 = U(m)$: $U(n) \times U(m) \times M \rightarrow M, ((U, V), z) \mapsto UzV^{-1}$. Assume that $n \geq m \geq 1$. The (equivariant) moment maps corresponding to these actions are given by $\Phi_1^\xi(z) = -\frac{1}{2} \text{Im tr}(\xi z \bar{z}^T)$ (for $\xi \in \mathfrak{u}(n)$) and $\Phi_2^\eta(z) = \frac{1}{2} \text{Im tr}(\eta \bar{z}^T z)$ (for $\eta \in \mathfrak{u}(m)$). Direct inspection shows that any level set of Φ_1 is exactly one $U(m)$ -orbit. Using standard facts from invariant theory and a theorem of G. Schwarz (see [Sch75], Thm. 1) one obtains that the space of pullbacks of smooth functions on \mathfrak{g}_1^* under Φ_1 coincides with the algebra of G_2 -invariant smooth functions on M , and thus by Lemma 2.2 with the centralizer of the Φ_2 -pullback of $C^\infty(\mathfrak{g}_2^*)$ in the Poisson algebra $C^\infty(M)$. The analogous statements hold if the roles of the two groups are reversed. Therefore, the symplectic Howe condition is satisfied.

In order to state the orbit correspondence of Thm. 2.8 explicitly, recall that any $z \in M$ can be written $z = U\Sigma V$ for

$$\Sigma = \Sigma(\sigma_1, \dots, \sigma_m) = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \in M,$$

where $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m \geq 0$ and $U \in U(n), V \in U(m)$ are determined up to an element of the stabilizer of Σ . The set $S = \{\sigma = (\sigma_1, \dots, \sigma_m) \mid \sigma_1 > \sigma_2 > \dots > \sigma_m > 0\}$ parametrizes thus all $U(n) \times U(m)$ -orbits of maximal dimension and

its closure \bar{S} all $U(n) \times U(m)$ -orbits. Identifying the adjoint and the coadjoint action of a unitary group via the usual trace form isomorphism ($\zeta \mapsto \text{tr}(\cdot \zeta)$) we get $\tilde{\Phi}_1(z) = \frac{i}{2} z \bar{z}^T$ and $\tilde{\Phi}_2(z) = -\frac{i}{2} \bar{z}^T z$ fulfilling $\Phi_1^\xi(z) = \text{tr}(\xi \cdot \tilde{\Phi}_1(z))$ and $\Phi_2^\eta(z) = \text{tr}(\eta \cdot \tilde{\Phi}_2(z))$. Denoting furthermore the diagonal element of $\mathfrak{u}(N)$ with entries $i(\lambda_1, \dots, \lambda_N)$ by $[\lambda_1, \dots, \lambda_N]$ or simply by $[\lambda]$ we have here the following orbit correspondence map:

$$\Lambda : \mathcal{O}_{[\frac{1}{2}\sigma_1^2, \dots, \frac{1}{2}\sigma_m^2, 0, \dots, 0]}^{U(n)} \mapsto \mathcal{O}_{[-\frac{1}{2}\sigma_1^2, \dots, -\frac{1}{2}\sigma_m^2]}^{U(m)} .$$

Observe that orbits of any dimension are included in this statement and no restriction to the principal orbit type is necessary. For the sake of brevity we write $[\alpha, 0]$ for $[\alpha_1, \dots, \alpha_m, 0, \dots, 0]$ in the sequel.

Now, we may apply Thm. 4.5 and obtain, since the prequantum line bundle L over M is holomorphically trivial,

$$\begin{aligned} \text{Hol}(M) = \Gamma_{\text{hol}}(M, L) &= \widehat{\bigoplus_{\alpha \in \bar{S} \text{ integral}}} \Gamma_{\text{hol}}(\mathcal{O}_{[\alpha, 0]}^{U(n)}, L_{[\alpha, 0]}) \otimes \Gamma_{\text{hol}}(\mathcal{O}_{[-\alpha]}^{U(m)}, L_{[-\alpha]}) \\ &\cong \widehat{\bigoplus_{\alpha \in \bar{S} \text{ integral}}} V_{[\alpha, 0]} \otimes (V_{[\alpha]})^* , \end{aligned}$$

where for $\lambda = (\lambda_1, \dots, \lambda_N)$ (with $\lambda_1, \dots, \lambda_n$ integral and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$), $V_{[\lambda]}$ is the irreducible $U(N)$ -representation associated to the orbit through $i(\lambda_1, \dots, \lambda_N)$ either by the orbit method or equivalently by the highest-weight procedure. The ensuing Howe duality map, $V_{[\alpha, 0]} \mapsto (V_{[\alpha]})^*$, is then essentially a restatement of the well-known $GL(n)$ - $GL(m)$ -duality (compare, e.g., Thm. 5.2.7 in [GW98]) on the space of holomorphic polynomials on M .

5.2 (G, G) on T^*G

Given a Lie group G , the left and right action of G on itself of course lift to its cotangent bundle T^*G . Both actions are free and proper, but this is not in general true for the product. Trivializing T^*G via the right action, the left resp. right action reads as follows: $(g, \alpha) \mapsto (hg, \text{Ad}^*(h)\alpha)$ and $(g, \alpha) \mapsto (gh^{-1}, \alpha)$, respectively. Furthermore, the moment maps are then given by $\Phi_1 : (g, \alpha) \mapsto \alpha$ and $\Phi_2 : (g, \alpha) \mapsto -\text{Ad}^*(g^{-1})\alpha$ for the left and right action, respectively. Obviously, the Φ_1 -fibres are right- G -orbits and the Φ_2 -fibres left- G -orbits. For connected G the symplectic Howe condition can easily be verified using Lemma 2.2; the orbit correspondence is $\Lambda : \mathcal{O}_\alpha \mapsto \mathcal{O}_{-\alpha}$.

Restricting now to the case that G is compact and connected, T^*G can be identified with the affine Kähler manifold $G^\mathbb{C}$ (see, e.g., Sect. 3 in [Hal97]) and the holomorphic geometric quantization $\text{Hol}(G^\mathbb{C})$ has the same $(G \times G)$ -finite vectors as the representation on $L^2(G)$. In this situation, Thm. 4.5 specializes thus to the Peter-Weyl theorem.

5.3 $(U(1), U(n))$ on $\mathbb{P}_n(\mathbb{C})$

The standard $U(n+1)$ -action on $\mathbb{P}_n(\mathbb{C})$ preserves the complex structure and the scaled Fubini-Study Kähler form $k\omega^{\text{FS}}$ ($k \in \mathbb{N} \setminus \{0\}$), and possesses the following equivariant moment map (for $\zeta \in \mathfrak{u}(n+1)$):

$$\Phi^\zeta([z]) = k \frac{i}{2\pi} \frac{\langle z, \zeta \cdot z \rangle}{\langle z, z \rangle}.$$

The injections $U(n) \rightarrow U(n+1)$, $A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ and $U(1) \rightarrow U(n+1)$, $e^{2\pi i \vartheta} \mapsto \begin{pmatrix} e^{2\pi i \vartheta} & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}$ yield a pair of commuting symplectic actions with induced moment maps. A study of invariant functions on $\mathbb{P}_n(\mathbb{C})$ shows that the action of the pair $(U(1), U(n))$ satisfies the symplectic Howe condition.

Fixing $\xi_0 = 2\pi i$ the moment map Φ_1 reads as $\Phi_1^{\xi_0} \cdot \xi_0^* \in \mathfrak{u}(1)^*$ and -identifying $\mathfrak{u}(1)^*$ with \mathbb{R} via the base consisting of ξ_0^* - one has that $\Phi_1(M) = [-k, 0]$. We also find that $\Phi_2(M) = U(n) \cdot S$, where $S = \{y \cdot \varphi_{11} \mid 0 \leq y \leq k\}$ is a global slice, with $\varphi_{11}(X) = \frac{i}{2\pi} X_{11}$ for $X \in \mathfrak{u}(n) \subseteq \text{Mat}(n \times n; \mathbb{C})$. The orbit correspondence Λ is then given by $x \mapsto U(n) \cdot ((x+k)\varphi_{11})$. Integral points in $\Phi_1(M)$ are simply points $m \in [-k, 0] \cap \mathbb{Z}$ with associated representation $e^{2\pi i \vartheta} \mapsto e^{2\pi i m \vartheta}$, whereas in $\Phi_2(M)$ an orbit $U(n) \cdot (y\varphi_{11})$ with $y \geq 0$ is integral if and only if $y \in \mathbb{N}$. Obviously, integrality of orbits in the sense of Def. 3.5 is preserved by the orbit correspondence map Λ .

For $k > 0$, the geometric quantization of $(\mathbb{P}_n(\mathbb{C}), k\omega^{\text{FS}})$ is given by the holomorphic sections module of the k -th power of the hyperplane bundle over $\mathbb{P}_n(\mathbb{C})$, isomorphic as a $U(n+1)$ -representation to $\mathbb{C}_k[z_0, z_1, \dots, z_n]$, the space of complex homogeneous polynomials of degree k in $n+1$ variables.

As a $(U(1) \times U(n))$ -representation this space decomposes as

$$\bigoplus_{d=0}^k \mathbb{C}_d[z_0] \otimes \mathbb{C}_{k-d}[z_1, \dots, z_n],$$

where for $l \in \mathbb{N}$, $\mathbb{C}_l[z_1, \dots, z_n]$ is the space of complex homogeneous polynomials of degree l in n variables, the $U(n)$ -representation associated to $U(n) \cdot (l\varphi_{11}) \in \Phi_2(M)/U(n)$, and $\mathbb{C}_d[z_0]$ realizes the $U(1)$ -representation on \mathbb{C} given by the character $e^{2\pi i \vartheta} \mapsto e^{2\pi i (-d)\vartheta}$ associated to $-d \in \Phi_1(M) \cap \mathbb{Z}$. The representation-theoretic Howe duality map of this $(U(1) \times U(n))$ -representation, $\mathbb{C}_d[z_0] \mapsto \mathbb{C}_{k-d}[z_1, \dots, z_n]$, is then clearly induced by the orbit correspondence map.

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References

- [Ada87] Jeffrey Adams. Coadjoint orbits and reductive dual pairs. *Adv. in Math.*, 63:138–151, 1987.
- [Dui96] Johannes J. Duistermaat. *The heat kernel Lefschetz fixed point formula for the spin^c Dirac operator*. Progress in Nonlinear Differential Equations and their Applications, 18. Birkhäuser Boston Inc., Boston, MA, 1996.
- [GS82] Victor Guillemin and Shlomo Sternberg. Geometric quantization and multiplicities of group representations. *Invent. Math.*, 67:515–538, 1982.
- [GS84] Victor Guillemin and Shlomo Sternberg. Multiplicity-free spaces. *J. Differential Geometry*, 19:31–56, 1984.
- [GW98] Roe Goodman and Nolan R. Wallach. *Representations and invariants of the classical groups*, volume 68 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1998.
- [Hal97] Brian C. Hall. Phase space bounds for quantum mechanics on a compact Lie group. *Comm. Math. Phys.*, 184(1):233–250, 1997.
- [How89] Roger Howe. Remarks on classical invariant theory. *Trans. Amer. Math. Soc.*, 313(2):539–570, 1989.
- [HW90] Alan T. Huckleberry and Tilmann Wurzbacher. Multiplicity-free complex manifolds. *Math. Ann.*, 286(1-3):261–280, 1990.
- [Kau67] Ludger Kaup. Eine Künnethformel für Fréchetgarben. *Math. Z.*, 97:158–168, 1967.
- [KKS78] D. Kazhdan, B. Kostant, and S. Sternberg. Hamiltonian group actions and dynamical systems of Calogero type. *Comm. Pure Appl. Math.*, 31(4):481–507, 1978.
- [KL97] Yael Karshon and Eugene Lerman. The centralizer of invariant functions and division properties of the moment map. *Illinois J. Math.*, 41(3):462–487, 1997.
- [Kos70] Bertram Kostant. Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*, pages 87–208. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.
- [OR04] Juan-Pablo Ortega and Tudor S. Ratiu. *Momentum maps and Hamiltonian reduction*, volume 222 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2004.
- [RN76] Léon Rosenfeld and Jens Rud Nielsen (eds.). *The Correspondence Principle (1918–1923)*, volume 3 of *Niels Bohr Collected Works*. North-Holland, Amsterdam, 1976.
- [Sch75] Gerald W. Schwarz. Smooth functions invariant under the action of a compact Lie group. *Topology*, 14:63–68, 1975.

- [Sja95] Reyer Sjamaar. Holomorphic slices, symplectic reduction and multiplicities of representations. *Ann. of Math. (2)*, 141(1):87–129, 1995.
- [SW59] Joseph H. Sampson and Gerard Washnitzer. A Künneth formula for coherent algebraic sheaves. *Illinois J. Math.*, 3:389–402, 1959.
- [Wei83] Alan Weinstein. The local structure of Poisson manifolds. *J. Differential Geom.*, 18(3):523–557, 1983.